Landau damping and coherent structures in narrow-banded 1+1 deep water gravity waves

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We study the modulational instability in surface gravity waves with random phase spectra. Starting from the nonlinear Schrödinger equation and using the Wigner-Moyal transform, we study the stability of the narrow-banded approximation of a typical wind-wave spectrum, i.e., the JONSWAP spectrum. By performing numerical simulations of the nonlinear Schrödinger equation we show that in the unstable regime, the nonlinear stage of the modulational instability is responsible for the formation of coherent structures. Furthermore, a Landau-type damping, due to the incoherence of the waves, whose role is to provide a stabilizing effect against the modulational instability, is both analytically and numerically discussed.

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I. INTRODUCTION

In many different fields of nonlinear physics, local nonlinear effects such as the modulational instability (MI) have played a very important role [1]. In plasmas, in the large-amplitude regime a nonlinear coupling between high-frequency Langmuir and low-frequency ion-acoustic waves takes place [2]. Under suitable physical conditions, the dynamics can be described by a nonlinear Schrödinger (NLS) equation and the modulational instability (MI) can be analyzed directly with this equation [3]. In nonlinear optics, the propagation of large amplitude electromagnetic waves produces a modification of the refractive index which, in turn, affects the propagation itself and makes possible the formation of wave envelopes. In the slowly varying amplitude approximation, this propagation is governed again by suitable NLS equations [4] and the MI plays a very important role [5]. The charged-particle beam dynamics in high-energy circular accelerating machines has been suitably described in terms of NLS for a complex wave function whose squared modulus is proportional to the beam density [6]. In this context, the so-called "coherent instability" due to the collective interaction of the beam with the surroundings [7], has been recently interpreted as the MI of the NLS equation [6].

For ocean gravity waves, the subject of this paper, the MI (also known as the Benjamin-Feir instability) has been discovered independently by Benjamin and Feir [8] and by Zakharov [9] in the 1960s. The instability predicts that in deep water a monochromatic wave is unstable under suitable small perturbations. This instability is well described by the NLS equation and has been recently addressed as responsible for the formation of freak waves [10–13].

While the role of the modulational instability for a monochromatic wave has been widely studied, its role in a continuous wave spectrum characterized by random phases has deserved less attention. In order to approach statistically, the nonlinear energy transfer processes involved, one is interested in finding a suitable *kinetic equation*. As far as ocean waves are concerned, this kinetic equation has been derived independently by Hasselmann [14,15] and by Zakharov [9,16]. Besides the quasi-Gaussian approximation [17], one of the major hypothesis required to derive the kinetic equation is that of homogeneity, i.e., $\langle A(k)A^*(k')\rangle = n(k)\delta(k-k')$, where A is a complex wave amplitude describing the envelope of the wave train, k and k' are wave numbers, $n(k)$ is the spectral density function, and brackets indicate ensemble averages. According to the kinetic equation, the energy is transferred in an irreversible manner only when four waves interact resonantly. Unfortunately, the Hasselmann-Zakharov theory is not able to predict the modulational instability because the latter results from a correlation between the carrier wave and the sideband perturbations and, moreover, it is not the result of an exact resonance. Nevertheless, if the hypothesis of homogeneity of the system is relaxed (correlation between different wave numbers is allowed), an improved kinetic equation can be derived which is able to show a random version of the Benjamin-Feir instability. For surface gravity waves, this improvement is contained in the pioneering work by Alber [18], followed by the works of Crawford *et al.* [19] and Janssen [20–22]. Independent of these works [18–20], a similar approach has been developed for the large-amplitude electromagnetic wave-envelope propagation in nonlinear media [24], for the quantumlike description of the longitudinal charged-particle beam dynamics in high-energy accelerating machines [25] and for the resonant interaction between an instantaneously produced disturbance and a partially incoherent Langmuir wave [26]. In all the above approaches, the basic idea is to transit from the configuration space, where the NLS equation governs the wave-envelope propagation, to the phase space, where an appropriate kinetic equation is able to show a random version of the MI. This has been done by using the mathematical tool provided by the Wigner-Moyal transform [27]. Consenquently, the governing kinetic equation is nothing but a von Neumann-Weyl-like equation [28].

In this paper, we outline the approach formulated in Refs. [18–20] and discuss the modulational instability for random wave spectra. In particular, we identify the values of the parameters of the JONSWAP spectrum (see, for example, Ref. [14]) for which the spectrum itself is unstable. Furthermore, within the framework of the theory developed in Refs. [18–20], we show that a phenomenon similar to the Landau damping [29] can be predicted for ocean gravity waves whose role is to provide a stabilizing mechanism against the MI. In particular, we show, both analytically and numerically, that this phenomenon is due to the incoherence of the background solution (random wave spectra). By performing numerical simulations of the NLS equation, we show that in the unstable regime coherent structures naturally appear in the space-time plane. We stress that our focus herein is not to attempt to model ocean waves but instead to study leading order effects using the simplest weakly nonlinear and dispersive model in deep water, i.e., the NLS equation. The results obtained are very similar to the ones recently given in the literature that have shown the existence of a Landau-type damping in the dynamics of small perturbations on a partially incoherent background, consisting of a constant amplitude and a stochastically varying phase, in nonlinear optics [24], in charged-particle beam dynamics [25] and plasma physics [26].

II. STATISTICAL DESCRIPTION OF THE NLS EQUATION FOR WATER WAVES

The NLS equation for water waves in infinite depth was derived under the hypothesis of small steepness and narrow-banded spectra for the first time by Zakharov in 1968 [9]. While this equation is not appropriate for describing the dynamics in the tail of the spectrum or in the inertial range (exact four-wave resonant interactions are forbidden), it should describe with satisfactory accuracy the dynamics around the spectral peak. In dimensional form, in a frame of reference moving with the group velocity, the equation reads

$$\frac{\partial A}{\partial t}+i\mu\frac{\partial^2 A}{\partial x^2}+i\nu|A|^2A=0, \tag{1}$$

where in deep water $\mu=\omega_0/8k_0^2$ and $\nu=\omega_0 k_0^2/2$ with ω_0 the carrier angular frequency and k_0 the respective wave number. The free surface elevation $\zeta(x,t)$ is related to the complex envelope $A(x,t)$ in the following way:

$$\zeta(x,t)=\mathrm{Re}[A(x,t)e^{i(k_0x-\omega_0 t)}]. \tag{2}$$

Equation (1) is the starting point for deriving the required kinetic equation. Following Alber [18], the Wigner-Moyal transform [27] can be applied directly to the NLS equation. This transform allows one to give a representation of a function $A(x,t)$ both in configuration x and in wave number k space:

$$n(x,k,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\langle A^*(x+y/2,t)A(x-y/2,t)\rangle e^{-iky}dy. \tag{3}$$

$n(x,k,t)$ is a second-order correlator. In order to derive an evolution equation for $n(x,k,t)$, we take the time-derivative of Eq. (3) and use the NLS equation to remove the time derivative of the complex envelope A in the right-hand side of Eq. (3). The nonlinear term in NLS will generate a fourth-order correlator [a term of the form $\langle A_1A_1^*A_1A_2^*\rangle$ where $A_1=A(x+y/2)$ and $A_2=A(x-y/2)$], therefore, a new variable is introduced. In order to proceed in the calculation, a closure that relates fourth- and second-order correlators must be introduced. This closure is achieved by introducing the quasi-Gaussian approximation $\langle A_1A_1^*A_1A_2^*\rangle =2\langle A_1A_2^*\rangle\langle A_1A_1^*\rangle$. The fourth-order correlator can be split as the product of the sum of second-order correlators, discarding the fourth-order cumulants. This procedure is well known for the statistical description of water waves [16] and of many other fields such as plasma physics [30]. The resulting kinetic equation is the following von Neumann-Weyl-like equation:

$$\frac{\partial n(x,k,t)}{\partial t}+2\mu k\frac{\partial n(x,k,t)}{\partial x}+4\nu\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!2^{2m+1}}\times\frac{\partial^{2m+1}\langle|A(x,t)|^2\rangle}{\partial x^{2m+1}}\frac{\partial^{2m+1}n(x,k,t)}{\partial k^{2m+1}}=0, \tag{4}$$

with

$$\langle|A(x,t)|^2\rangle=\int_{-\infty}^{+\infty}n(x,k,t)dk. \tag{5}$$

If only the first term in the infinite sum is considered, the equation reduces to the Vlasov-Poisson equation in plasma physics that is well known to describe the Landau damping phenomenon [29]. This damping is due to the interactions of resonant electrons of the system; the theory predicts that the rate of decay of the wave energy is proportional to the first derivative of the equilibrium distribution function of the electrons.

A. Stability of wave spectra

In order to study the stability of wave spectra, a standard linear stability analysis of the von Neumann-Weyl-like equation is performed: we let the distribution function $n(x,k,t)$ be expressed in terms of an equilibrium distribution $n_0(k)$ plus a small perturbation,

$$n(x,k,t)=n_0(k)+n_1(x,k,t) \tag{6}$$

with $n_1(x,k,t)\ll n_0(k)$. After substituting Eq. (6) into Eq. (4) and linearizing we obtain the following equation for the perturbation:

$$\frac{\partial n_1(x,k,t)}{\partial t}+2\mu k\frac{\partial n_1(x,k,t)}{\partial x}+4\nu\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!2^{2m+1}}\times\frac{\partial^{2m+1}\langle|A(x,t)|^2\rangle}{\partial x^{2m+1}}\frac{\partial^{2m+1}n_0(k)}{\partial k^{2m+1}}=0. \tag{7}$$

We then look for solutions of the form

$$n_1(x,k,t)=\tilde{n}_1(k)e^{i(Kx-\Omega t)}. \tag{8}$$

After standard algebra, the following implicit form of the dispersion relation is obtained (see also, Refs. [24–26]):

$$1+\frac{\nu}{\mu}\int\frac{n_0(k+K/2)-n_0(k-K/2)}{K(k-\Omega/(2\mu K))}dk=0, \tag{9}$$

where $n_0(k)$ is the homogeneous envelope spectrum. Once the equilibrium solution n_0 is given, Eq. (9) represents the dispersion for the perturbation.

We now look for an equilibrium solution n_0 for sea surface gravity waves. According to experimental works conducted more than 25 years ago (see, for example, Ref. [14]), it has been found that the spectrum for the free surface elevation ζ is well approximated by the JONSWAP spectrum (we give its form in wave-number space using the dispersion relation in infinite water depth $\omega=\sqrt{gh}$):

$$P(k)=\frac{\alpha}{2k^3}e^{-(3/2)[k_0/k]^2}\gamma^{\exp[-(\sqrt{k}-\sqrt{k_0})^2/2\delta^2 k_0]}, \tag{10}$$

with α, γ, and δ constants (δ is usually set to 0.07, while α and γ depend on the state of the ocean). As α and γ are increased, the wave amplitude and, therefore, the nonlinearity of the wave train increases. γ rules also the spectral width: for large values of γ the spectrum becomes more narrow banded. Since we are interested in the dynamics around the peak of the spectrum, as is done in Ref. [21], we consider a second-order Taylor expansion of $P(k)$ around the peak k_0. It turns out that the spectrum in Ref. (10) reduces to the following Lorentzian one:

$$P(k)=\frac{H_s^2}{16\pi}\frac{p}{p^2+(k-k_0)^2}, \tag{11}$$

where

$$p=\sqrt{\frac{8k_0^2\delta^2}{24\delta^2+\ln(\gamma)}} \quad \text{and} \quad H_s=4\sqrt{\pi\frac{\alpha\gamma p}{2E^{3/2}k_0^3}}. \tag{12}$$

H_s is the significant wave height calculated as four times the standard deviation of the wave field and p corresponds exactly to the half-width at half-maximum of the spectrum. Note that the spectrum in Eq. (11) or in Eq. (10) is the spectrum for the free surface ζ and not for the envelope A. Nevertheless, it can be shown [19] that for a symmetric spectrum $P(k)$ of the surface elevation, the spectrum for the complex envelope is given by $n_0(k)=4P(k+k_0)$, therefore, a factor of 4 must be taken into account. Solving Eq. (9) with respect to Ω, we obtain the following dispersion relation:

$$\Omega=K(\sqrt{K^2\mu^2-H_s^2\nu\mu}-2i\mu p). \tag{13}$$

Positive complex roots of Eq. (13) result in a growing instability of the perturbation: if $K^2<H_s^2\nu/\mu$, the first term on the right-hand side is responsible for the MI (note that in the limit as $p\to 0$, the dispersion relation (13) gives the Benjamin-Feir instability). The last term on the right-hand side has a stabilizing effect and plays the same role played by the Landau damping in plasma physics [29], i.e., a damping of the perturbation. There is a competition between exponential growth and damping of the perturbation that depends on the parameters α and γ of the Lorenzian (or JONSWAP) spectrum. If $p>\mathrm{Im}[\sqrt{+K^2/4-H_s^2\nu/(4\mu)}]$ the damping dominates the MI, the opposite will occur if $p<\mathrm{Im}[\sqrt{K^2/4-H_s^2\nu/(4\mu)}]$. In Fig. 1 we show the marginal stability curve in the α-γ plane in the limit of $K\to 0$. Spectra with higher values of α and γ are more likely to show the MI. In the following section, we will perform numerical simulations of the NLS equation in order verify the result from the dispersion relation (13) and study the effect of the instability in physical space.

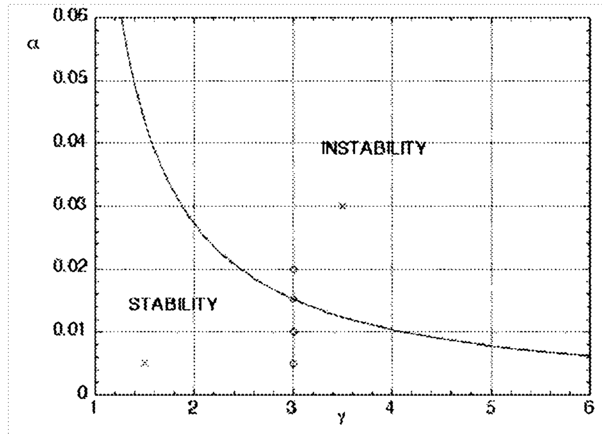

FIG. 1. Instability diagram in the α-γ plane. α and γ are nondimensional variables.

B. Analogy with plasma physics

Let us now present some remarks about the physical origin of the Landau-type damping predicted above. In plasma physics, this damping, for instance, is caused by resonant interactions between a plasma wave and the electrons. By denoting with v the single-particle velocity and with ω and κ the frequency and wave number of the electron plasma wave, respectively, the Landau theory, based on the Vlasov kinetic equation, clearly shows that the decay rate of the wave energy is proportional to the first derivative of the equilibrium distribution function $\rho_0(v)$ of the electrons. Typically, the shape of $\rho_0(v)$ is such that $d\rho_0(v=\omega/\kappa)/dv<0$, which implies that there are more particles with $v<\omega/\kappa$ (which gain energy from the wave) than with $v>\omega/\kappa$ (which give energy to the wave). This statistical circumstance leads to a net damping of the plasma wave. This is usually referred as to ''weak Landau damping.'' Additionally, as the thermal dispersion of the electrons of the plasma becomes negligible (for example, the equilibrium distribution becomes more and more sharp), the Landau damping becomes more and more weak. In principle, a cold plasma, whose thermal distribution corresponds to a δ-function, does not exhibit the phenomenon of Landau damping.

On the other hand, we observe that, in the limiting case for $K\ll k$ dispersion relation (9) becomes

$$1+\frac{\nu}{\mu}\int\frac{dn_0/dk}{k-\Omega/(2\mu K)}dk=0, \tag{14}$$

where we have used the approximation

$$\frac{n_0(k+K/2)-n_0(k-K/2)}{K}\approx\frac{dn_0}{dk}.$$

Given the full similarity between the dispersion relation (14) and the one found by Landau [29], the physical origin of the stabilizing effect predicted in the present paper may be described by using an analogy with the phenomenon of Landau damping. In fact, the above stabilizing effect can be attributed to the "nonmonochromatic" character of the Wigner spectrum of the surface gravity waves. Similar to the standard Landau damping, where the electrons interact individually with a linear plasma wave and statistically produce a net transfer of energy from the wave to the particles, the gravity wave train interacts with the perturbation and produces a transfer of energy between wave numbers which is more significant around $k=\Omega/2\mu K$.

Furthermore, it is worth noting that, on the basis of the above physical interpretation, the stabilizing effects that come out also from the more general dispersion relation (9), may be thought of as an extension, to arbitrary wave numbers, of the Landau-type damping of an ensemble of incoherent surface gravity waves. However, this effect is not the analog of the weak Landau damping and cannot be predicted with a Vlasov-like equation. In fact, for arbitrary K, and for the broad-band spectrum (11), all the terms of the von Neumann-Weyl kinetic-like equation (4) contribute to the dispersion relation (13).

III. NUMERICAL SIMULATIONS AND DISCUSSION

Numerical simulations of Eq. (1) have been computed using a standard pseudospectral Fourier method. Initial conditions for the free surface elevation $\zeta(x,0)$ have been constructed as the following random process [31]:

$$\zeta(x,0)=\sum_{n=1}^{N} C_n\cos(k_n x-\phi_n), \tag{15}$$

where ϕ_n are uniformly distributed random numbers on the interval $(0,2\pi)$, and $C_i=\sqrt{2P(k_i)\Delta k_i}$, where $P(k_i)$ is the discretized spectrum given in Eq. (11). The Hilbert transform is used in order to convert the free surface ζ to the complex envelope variable A of the NLS. The spectrum of the complex envelope A is nothing other than the unperturbed homogeneous solution $n_0(k)$. The dominant wave number for the numerical simulation was selected to be $k_0=0.1\ \mathrm{m}^{-1}$. This last choice is not restrictive: the parameters that rule the dynamics of the spectrum are its width and the steepness which, once k_0 is fixed, are univocally determined by α and γ.

In order to investigate the effects of the instability, we have chosen two different initial conditions characterized by values of α and γ such that the dispersion relation (13) predicts, respectively, instability and stability. We have considered values in the α-γ plane that are far away from the marginal stability curve (see crosses in Fig. 1). For the unstable case, we have considered $\gamma=3.5$ and $\alpha=0.03$ and for the stable one we have taken $\gamma=1.5$ and $\alpha=0.005$

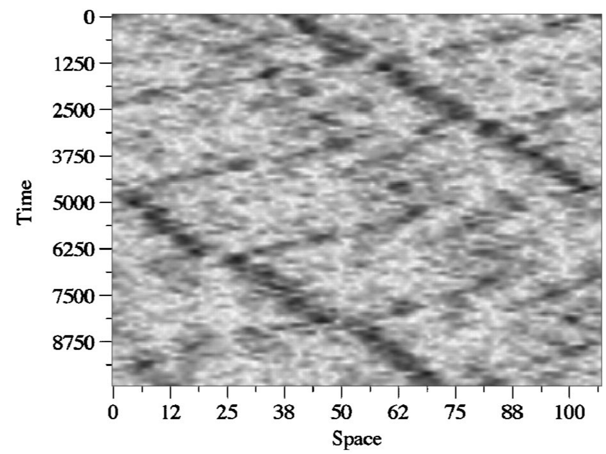

FIG. 2. $|A(x,t)|$ from numerical simulation of the NLS. The initial condition is characterized by a Lorenzian spectrum with $\gamma=3.5$ and $\alpha=0.03$. A coherent structure is evident in the x-t plane. Space and time have been scaled, respectively, with k_0 and $\omega_0=\sqrt{gk_0}$.

We now start the discussion of the numerical results by showing in Fig. 2 the evolution of $|A(x,t)|$ in the x-t plane for the unstable case. How is this instability manifested? From Fig. 2, we note the presence of a "coherent structure," i.e., a structure (oblique darker zones in the x-t plane) that persists in the presence of nonlinear interactions and maintains statistically its shape and velocity during propagation (note that periodic boundary conditions are used). Every random realization with the same values of α and γ shows similar results even though the resulting coherent structures may have different velocity and amplitude. If nonlinearity is increased, more than one coherent structure may appear in the x-t plane. The nonlinear stage of MI is therefore responsible for the formation of such coherent structures in the x-t plane. Indeed it is possible to show that the NLS equation has periodic solutions such as breathers or unstable modes [13,32]. These solutions, which are the result of a linear instability, are nevertheless very robust. Moreover they can grow up to more than three times the initial unperturbed solution and, therefore, have also been addressed as simple models for freak waves [10–13]. In contrast to solitons that have constant amplitude in time, these unstable modes are characterized by a continuous exchange of energy among the Fourier modes. The energy is transferred from one mode to another and back again: the process is completely reversible and, therefore, coherent structures persist in physical space. We stress that these kinds of solutions appear naturally from initial conditions with random phases. The striking result here is that even if initial conditions are completely random, the nonlinear interactions generate a strong correlation among wave numbers resulting in coherent structures embedded in a random wave field.

We now consider the stable case. We again show the space-time evolution of $|A(x,t)|$ in Fig. 3. The x-t plane appears as a random field and there is no evidence of any structure that survives for long periods of time. Numerical

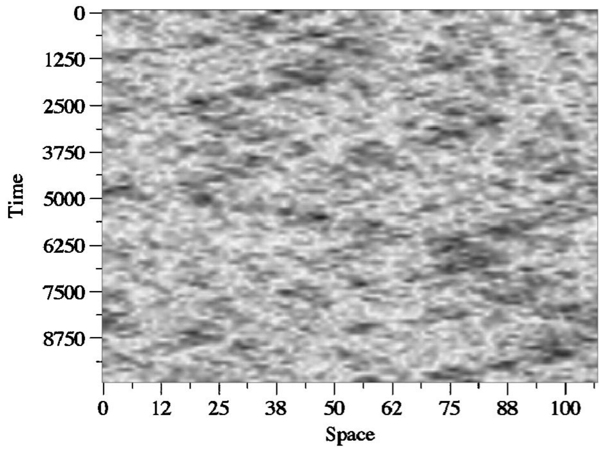

FIG. 3. $|A(x,t)|$ from numerical simulation of the NLS. The initial condition is characterized by a Lorenzian spectrum with $\gamma = 1.5$ and $\alpha = 0.005$. The field in the x-t plane appears to be random without any evidence of coherent structure. Space and time have been scaled, respectively, with k_0 and $\omega_0 = \sqrt{gk_0}$.

simulations with initial conditions characterized by the same values of α and γ but with different random phases are in accordance with the results just shown and are not here reported.

We will now answer to the following two natural questions: How sharp is the transition region in Fig. 1? How do coherent structures develop from random phase initial conditions? We have performed a number of numerical simulations with initial conditions characterized by values of α and γ that are close to the marginal stability curve. In particular, here we report numerical simulations that have been obtained by setting $\gamma = 3$ and $\alpha = 0.005$, 0.01, 0.0153, and 0.02. Circles in Fig. 1 are located where these last numerical simulations have been performed. As is clear from the figure, points are selected in order to cross the margial stability curve: we move accross the stability curve by changing the value of α. In Figs. 4(a)–4(d) we show the evolution of $|A(x,t)|$ in the x-t plane, respectively, for $\gamma = 3$ and $\alpha = 0.005$, 0.01, 0.0153, and 0.02. Figures 4(a) and 4(b) are the result of the evolution of initial conditions characterized by values of α and γ for which the theory predicts a stable regime, see Fig. 1. While in Fig. 4(a) there is no clear evidence of a coherent structure, in Fig. 4(b) a darker coherent region has already developed. We recall that larger value of α implies larger waves (the steepness is increased) and, therefore, an increase in nonlinearity of the initial conditions. The effect of nonlinearity is such that the uncorrelated wave numbers at time $t = 0$ sec develop some correlations and as a result coherent structures naturally are formed. As we increase α, Figs. 4(c) and 4(d), coherent structures become more and more well defined.

From the numerical simulations just shown, it seems that coherent structures can appear from initial conditions characterized by values of α and γ taken below the marginal stability curve. This result is not so surprising: the marginal stability curve has been recovered via a linear stability analysis of the kinetic equation which is the result of a statistical approach to the NLS equation. However, in the natural long-time evolution of a nonlinear wave train, perturbations are in general not small. Consequently, perturbations in the simulations should evolve according to the governing equations. Our observation is consistent with some very recent results obtained by Janssen [22]; he was the first one to point out that the broadening of the spectrum in numerical simulations of the NLS and Zakharov equation starts for values of the steepness and spectral width that are lower with respect to the one predicted by the nonhomgeneous theory (see also Ref. [23]). In order to explain this result, he has proposed a modification of the Hasselmann-Zakharov kinetic equation by taking into account also nonresonant interactions [22].

IV. CONCLUSIONS AND REMARKS

In this paper, we have studied the stability of random wave spectra for surface gravity waves in $(1+1)$ dimension. Theoretical results from a Wigner approach on the NLS equation are compared with direct numerical simulations of the NLS equation. One interesting result of this study concerns the effect of the instability in physical space: numerical simulations show that, starting with a random wave field, coherent structures naturally develop as long as the initial conditions have sufficient energetics (large α) and are narrow banded (large γ). The theory developed by using the Wigner-Moyal transform allows one to isolate the region in the α-γ plane where those structure are more likely to appear. The transition region predicted by the theory does not appear to be sharp. Coherent structures appear for values of α and γ lower than predicted by the theory. Our numerical simulations are in one space dimension and moreover the spectrum that we have used in the theory and numerical simulations is a narrow-banded approximation of the

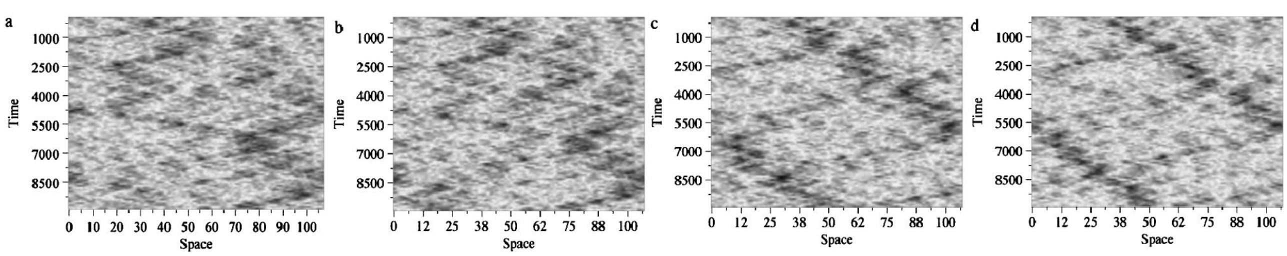

FIG. 4. $|A(x,t)|$ from numerical simulation of the NLS. The initial condition is characterized by a Lorenzian spectrum with $\gamma = 3$ and $\alpha = 0.005$ (a), 0.01 (b), 0.0153 (c), and 0.02 (d). Space and time have been scaled, respectively, with k_0 and $\omega_0 = \sqrt{gk_0}$.

JONSWAP spectrum. Needless to say, the theory cannot be taken as quantitative. Nevertheless, the marginal stability curve could give a first qualitative indication of unstable spectra in realistic conditions in infinite water depth (values of α and γ here considered are typical of ocean waves). Many physical questions remain open. For example, it would be interesting to investigate the case of a two-dimensional wave field. It is well known that the NLS in $(2+1)$ dimension is not integrable and the dynamics of coherent structures is still far from being understood. Numerical simulations with the fully nonlinear Euler equations are also under consideration in order to extend the validity of these results.

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